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New aspects in polygroup theory

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Abstract

The aim of this paper is to compute the commutativity degree in polygroup's theory, more exactly for the polygroup P_G and for extension of polygroups by polygroups, obtaining boundaries for them. Also, we have analyzed the nilpotencitiy of $\mathcal{A}[\mathcal{B}]$, meaning the extension of polygroups $\mathcal{A}$ and $\mathcal{B}$.

1 Introduction

The polygroups theory represents a particular class from the hypergroup theory. This theory is detailed in the book of Davvaz, "Polygroup Theory and Related Systems" see [4]. We choose this class because it is similar to group theory and we founded a few similarities but and differences between these two theories.

Definition 1. *A polygroup is a system $\varphi =< P, \cdot, e, ^{-1} >$, where $e \in P$, $^{-1}$ is a unitary operation on P and " $\cdot$ " : $P \times P \to \mathcal{P}^*(P)$. In the following, the next axioms hold for all x, y, $z \in P$:*

i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;

ii) $e \cdot x = x \cdot e = x$;

iii) $x \in y \cdot z$, *implies* $y \in x \cdot z^{-1}$ *and* $z \in y^{-1} \cdot x$.

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Also, from the above axioms, it is obtaine:

$$\begin{aligned} e &\in x \cdot x^{-1} \cap x^{-1} \cdot x; e^{-1} = e, \\ \left(x^{-1}\right)^{-1} &= x, (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}. \end{aligned}$$

2 Commutativity degree in polygroup theory

The aim of this section is to compute the commutativity degree for polygroup P_G and to find a connection between the results from group theory and from polygroup theory. This notion, was studied by Azam Hokmabadi, Fahimeh Mohammadzadeh and Elaheh Mohammadzade, see [7] presented in the 6^{th} International Group Theory Conference, 2014. In this paper, the definition of commutativity degree has a similar form, but we don't using the heart of a polygroup.

Definition 2. *Let $< P, \cdot, e, ^{-1} >$ be a polygroup. The commutativity degree of polygroup P, notice by $d(P)$ has the next form:*

$$d(P) = \frac{|\{(a, b) \in P^2 |\ a \cdot b = b \cdot a\}|}{|P|^2}.$$

Remark 3. *The set $\{(a, b) \in P^2 |\ a \cdot b = b \cdot a\}$ is notice by $c(P)$.*

Example 4. *Let $P = \{e, a, b, c\}$ and let $< P, \cdot, e, ^{-1} >$ be a non-commutative polygroup, where " $\cdot$ " is define thus*

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	a	P	c
b	b	$\{e, a, b\}$	b	$\{b, c\}$
c	c	$\{a, c\}$	c	P

In this case, the commutativity degree of polygroup P, is

$$d(P) = \frac{10}{16} = \frac{5}{8}.$$

Proposition 5. *Let $< P_1, \cdot, e_1, ^{-1} >$ and $< P_2, *, e_2, ^{-1} >$ be two polygroups. $P_1 \times P_2$ equipped with the usual direct hyperproduct*

$$" \circ " : (P_1 \times P_2) \times (P_1 \times P_2) \to P_1 \times P_2,$$

$$(x_1, y_1) \circ (x_2, y_2) = \{(x, y) \mid x \in x_1 \cdot x_2,\ y \in y_1 * y_2\}$$

becomes a polygroup.

Proposition 6. *Let* $< P_1, \cdot, e_1, ^{-1} >$ *and* $< P_2, *, e_2, ^{-1} >$ *be two polygroups. The next relation holds*

$$d\left(P_1 \times P_2\right) = d(P_1)d(P_2).$$

Proof. The amount

$$\frac{|\{\, (x_1, y_1) \times (x_2, y_2) \in (P_1 \times P_2)^2 \mid\ (x_1, y_1) \circ (x_2, y_2) = (x_2, y_2) \circ (x_1, y_1)\, \}|}{|P_1 \times P_2|^2}. \tag{1}$$

represents the commutativity degree of $P_1 \times P_2$. So, the expression

$$(x_1, y_1) \circ (x_2, y_2) = (x_2, y_2) \circ (x_1, y_1) \tag{2}$$

is equivalent with

$$\begin{aligned} \{(x, y) \quad &\in \quad P_1 \times P_2 | x \in x_1 \cdot x_2 = x_2 \cdot x_1, y \in y_1 * y_2 = y_2 * y_1\} \\ &= \quad \{x \in P_1 |\ x \in x_1 \cdot x_2 = x_2 \cdot x_1\}\{y \in P_2 |\ y \in y_1 * y_2 = y_2 * y_1\} \\ &= \quad c\left(P_1\right) c\left(P_2\right). \end{aligned}$$

$$P_1 \times P_2 = \{(x, y)\,| x \in P_1, y \in P_2\} = \{x, x \in P_1\}\{y, y \in P_2\},$$

follows that

$$|P_1 \times P_2| = |P_1||P_2|.$$

Therefore,

$$d(P_1 \times P_2) = \frac{|c\left(P_1 \times P_2\right)|}{|P_1 \times P_2|^2} = \frac{|c\left(P_1\right)||c\left(P_2\right)|}{|P_1 \times P_2|^2}.$$

In conclusion,

$$d(P_1 \times P_2) = \frac{|c\left(P_1\right)|}{|P_1|^2} \frac{|c\left(P_2\right)|}{|P_2|^2} = d(P_1)d(P_2),$$

□

Example 7. *Let sets* $P_1 = \{0, 1\}$, $P_2 = \{e, a, b, c\}$ *and let* $< P_1, \cdot, e, ^{-1} >$, $< P_2, *, e', ^{-1} >$ *be two polygroups, where* " $\cdot$ " *şi* " $*$ " *are define thus:*

$\cdot$	0	1
0	0	1
1	1	0

and

$*$	e	a	b	c
e	e	a	b	c
a	a	a	P_2	c
b	b	$\{e, a, b\}$	b	$\{b, c\}$
c	c	$\{a, c\}$	c	P_2

We notice

$$\alpha_i^j = (x_i, y_j), i \in \{1,2\},\ j \in \{1,2,3\},$$

where x_i *and* y_j*, represents of component* i *from* P_1 *and* y_j *represents of component* j *from* P_2*. The product polygroup* $P_1 \times P_2$ *has the next form.*

$\circ$	α_1^1	α_1^2	α_1^3	α_1^4
α_1^1	α_1^1	α_1^2	α_1^3	α_1^4
α_1^2	α_1^2	α_1^2	$\left\{\begin{array}{c}\alpha_1^i,\\ i=\overline{1,4}\end{array}\right\}$	α_1^4
α_1^3	α_1^3	$\left\{\begin{array}{c}\alpha_1^i,\\ i=\overline{1,3}\end{array}\right\}$	α_1^3	$\{\alpha_1^3, \alpha_1^4\}$
α_1^4	α_1^4	$\{\alpha_1^2, \alpha_1^4\}$	α_1^4	$\left\{\begin{array}{c}\alpha_1^i,\\ i=\overline{1,4}\end{array}\right\}$
α_2^1	α_2^1	α_2^2	α_2^3	α_2^4
α_2^2	α_2^2	α_2^2	$\left\{\begin{array}{c}\alpha_2^i,\\ i=\overline{1,4}\end{array}\right\}$	α_2^4
α_2^3	α_2^3	$\left\{\begin{array}{c}\alpha_2^i,\\ i=\overline{1,3}\end{array}\right\}$	α_2^3	$\{\alpha_2^3, \alpha_2^4\}$
α_2^4	α_2^4	$\{\alpha_2^2, \alpha_2^4\}$	α_2^4	$\left\{\begin{array}{c}\alpha_2^i,\\ i=\overline{1,4}\end{array}\right\}$

and

$\circ$	α_2^1	α_2^2	α_2^3	α_2^4
α_1^1	α_2^1	α_2^2	α_2^3	α_2^4
α_1^2	α_2^2	α_2^2	$\{\alpha_2^i,\ i=\overline{1,4}\}$	α_2^4
α_1^3	α_2^3	$\{\alpha_2^i, i=\overline{1,3}\}$	α_2^3	$\{\alpha_2^3, \alpha_2^4\}$
α_1^4	α_2^4	$\{\alpha_2^2, \alpha_2^4\}$	α_2^4	$\{\alpha_2^i,\ i=\overline{1,4}\}$
α_2^1	α_1^1	α_1^2	α_1^3	α_1^4
α_2^2	α_1^2	α_1^2	$\{\alpha_1^i,\ i=\overline{1,4}\}$	α_1^4
α_2^3	α_1^3	$\{\alpha_1^i, i=\overline{1,3}\}$	α_1^3	$\{\alpha_1^3, \alpha_1^4\}$
α_2^4	α_1^4	$\{\alpha_1^2, \alpha_1^4\}$	α_1^4	$\{\alpha_1^i,\ i=\overline{1,4}\}$

The commutativity degree is

$$d(P_1 \times P_2) = \frac{40}{64} = \frac{5}{8} \cdot 1 = d(P_1) \cdot d(P_2).$$

Let $(G, \cdot)$ be a group and $P_G = G \cup \{a\}$, where $a \notin G$. It is define on P_G, the hyperoperation " $\circ$ " as follows

$$\begin{array}{lll}(1) & : & a \circ a = e; \\ (2) & : & e \circ x = x \circ e = x, \forall x \in P_G; \\ (3) & : & a \circ x = x \circ a = x, \forall x \in P_G \backslash \{e, a\}; \\ (4) & : & x \circ y = x \cdot y, \forall (x, y) \in G^2,\ y \neq x^{-1}; \\ (5) & : & x \circ x^{-1} = \{e, a\},\ \forall x \in P_G \backslash \{e, a\}.\end{array}$$

Proposition 8. *If G is a group, then $< P_G, \circ, e, ^{-1} >$ is a polygroup.*

Corolar 9. *Let $(G, \cdot)$ be a group. The polygroup P_G is nilpotent, if and only if G is a nilpotent group.*

Proposition 10. *If $(G, \cdot)$ is a finit group, with $|G| = n$, $n \in \mathbb{N}^*$, then*

$$d\left(P_G\right) = \frac{n^2 d(G) + 2n + 1}{\left(n+1\right)^2}. \tag{3}$$

Proof. We define, the commutativity degree of polygroup P_G as follows

$$d(P_G) = \frac{\left|\{(x, y) \in P_G^2 |\ x \circ y = y \circ x\}\right|}{|P_G|^2}. \tag{4}$$

Let

$$\begin{array}{lll}A_1 & = & \{(x, y) \in G^2,\ y \neq x^{-1}\},\ A_2 = \{(x, y) \in G^2,\ y = x^{-1}\}, \\ A_3 & = & \{(a, y),\ y \in G\},\ A_4 = \{(x, a),\ x \in G,\ y = a\},\ A_5 = \{(a, a)\}.\end{array}$$

We observe that

$$P_G \times P_G = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5, \tag{5}$$

with

$$A_i \cap A_j = \varnothing,\ \forall\ i \neq j. \tag{6}$$

According to (5) and (6), the above expression, could be written thus

$$\begin{aligned}\left|\{(x, y) \in P_G^2 |\ x \circ y = y \circ x\}\right| &= \sum_{i=1}^{5} |\{(x, y) \in A_i |\ x \circ y = y \circ x| \\ &= n^2 d(G) + n + n + 1 = n^2 d(G) + 2n + 1.\end{aligned}$$

So,

$$d\left(P_G\right) = \frac{n^2 d(G) + 2n + 1}{\left(n+1\right)^2}.$$

$\square$

Example 11. *If $G = D_3$, then $P_G = G \cup a$, $a \notin D_3$. The commutativity degree of G, is $d(G) = \frac{1}{2}$.*

$\circ$	e	ρ	ρ^2	σ	$\rho\sigma$	$\rho^2\sigma$	a
e	e	ρ	ρ^2	σ	$\rho\sigma$	$\rho^2\sigma$	a
ρ	ρ	ρ^2	$\{e,a\}$	$\rho\sigma$	$\rho^2\sigma$	σ	ρ
ρ^2	ρ^2	$\{e,a\}$	ρ	$\rho^2\sigma$	σ	$\rho\sigma$	ρ^2
σ	σ	$\rho^2\sigma$	$\rho\sigma$	$\{e,a\}$	ρ^2	ρ	σ
$\rho\sigma$	$\rho\sigma$	σ	$\rho^2\sigma$	ρ	$\{e,a\}$	ρ^2	ρ
$\rho^2\sigma$	$\rho^2\sigma$	$\rho\sigma$	σ	ρ^2	ρ	$\{e,a\}$	$\rho^2\sigma$
a	a	ρ	ρ^2	σ	$\rho\sigma$	$\rho^2\sigma$	e

$$d(P_G) = \frac{31}{49} = \frac{6^2 \cdot \frac{1}{2} + 2 \cdot 6 + 1}{7^2}.$$

Remark 12. **1**. *$d(P_G) \geq d(G)$, for all group G;*

2. *If G is an abelian group, then P_G is a commutative polygroup.*

3. *According to the above example, it is observed that there is a non commutative polygroup P_G with commutativity degree more than $\frac{5}{8}$, what in group theory does not happend.*

In what follows, we determine a bounded for polygroup P_G, which depends to $d(G)$.

Proposition 13. *If G is a group with $|G| = n$, then*

$$d(G) \leq d(P_G) \leq \frac{d(G)+3}{4}.$$

Proof. Let G be a group, with $|G| = n$. The first inequality is obvious, from Remark 12 and for second inequality, we make some elementary calculus and we obtain

$$(d(G)-1)(3n^2-2n-1) \leq 0, \forall n \geq 1.$$

It is true, because $d(G) \in (0,1]$ and $3n^2 - 2n - 1 = (n-1)(3n+1) \geq 0, \forall n \geq 1$. $\square$

Proposition 14. *P_G is a commutative polygroup if and only if $d(P_G) > \frac{29}{32}$.*

Proof. If P_G is commutative polygroup, follows that $d(P_G) = 1 > \frac{29}{32}$.

Inverse, if $d(P_G) > \frac{29}{32}$, then

$$\begin{aligned} \frac{n^2 d(G) + 2n + 1}{(n+1)^2} &> \frac{29}{32}, \text{ equivalent} \\ n^2(32d(G) - 29) + 6n + 3 &> 0, \text{ for all } n \geq 2. \end{aligned}$$

If G is abelian group, then $d(G) = 1$ and inequality is true.

If G is a non abelian group, then $d(G) < \frac{5}{8}$, so

$$n^2(32d(G) - 29) + 6n + 3 < -9n^2 + 6n + 3 < 0, \text{ for all } n \geq 2.$$

In this situation, the inequality doesn't holds.

In conclusion, P_G is a commutative polygroup if and only if $d(P_G) > \frac{29}{32}$. $\square$

3 Extension of polygroups by polygroups

The purpose of this section is to determine the commutativity degree of extension polygroups by polygroups and to find a connection with the commutativity degrees of the two polygroups which form the extension. Let $\mathcal{A} = < A, \cdot, e, ^{-1} >$ and $\mathcal{B} = < B, \cdot, e, ^{-1} >$ be two polygroups whose elements have been renamed so that încât $A \cap B = \{e\}$. A new system $\mathcal{A}[\mathcal{B}]$ called the extension of $\mathcal{A}$ by $\mathcal{B}$ is formed in the following way:

$$\mathcal{A}[\mathcal{B}] = < M, *, e, ^I >,$$

where

$$M = A \cup B, e^I = e, x^I = x^{-1}, e * x = x * e = x, \text{ for all } x \in M;$$

and for all x, $y \in M \backslash \{e\}$.

$$x * y = \begin{cases} x \cdot y & \text{if } x,\ y \in A \\ x & \text{if } x \in B,\ y \in A \\ y & \text{if } x \in A,\ y \in B \\ x \cdot y & \text{if } x, y \in B,\ y \neq x^{-1} \\ x \cdot y \cup A & \text{if } x, y \in B,\ y = x^{-1} \end{cases} \tag{7}$$

In this case, $\mathcal{A}[\mathcal{B}]$ is a polygroup which is called the extension of $\mathcal{A}$ by $\mathcal{B}$.

We consider $A = \{e, a_1, a_2, ..., a_{n-1}\}$ şi $B = \{e, b_1, b_2, ..., b_{m-1}\}$, where n, $m \in \mathbb{N}^*$. We can represent the operation " $*$ " through next table:

$*$	e	a_1	...	a_{n-1}	b_1	...	b_i	...	b_{m-1}
e	e	a_1	...	a_{n-1}	b_1	...	b_i	...	b_{m-1}
a_1	a_1	$a_1 a_1$	...	$a_1 a_{n-1}$	b_1	...	b_i	...	b_{m-1}
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	...	$\vdots$	...	$\vdots$
a_{n-1}	a_{n-1}	$a_{n-1} a_1$	...	$a_{n-1} a_{n-1}$	b_1	...	b_i	...	b_{m-1}
b_1	b_1	b_1	...	b_1	$b_1 b_1$	...	$b_1 b_i \cup A$	...	$b_1 b_{m-1}$
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	...	$\vdots$	...	$\vdots$
b_i	b_i	b_i	...	b_i	$b_i b_1 \cup A$	...	$b_i b_i$	...	$b_i b_{m-1}$
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	...	$\vdots$	...	$\vdots$
b_{m-1}	b_{m-1}	b_{m-1}	...	b_{m-1}	$b_{m-1} b_1$	...	$b_{m-1} b_i$	...	$b_{m-1} b_{m-1}$

Without loss generality, we suppose that $b_i = b_1^{-1}$. For each element b_j, it is exists unique b_k, such that $b_j = b_k^{-1}$ with $i,\ j,\ k \in \overline{1, m-1}$.

The commutativity degree of polygroup $\mathcal{A}[\mathcal{B}]$ it is define thus:

$$d(\mathcal{A}[\mathcal{B}]) = \frac{\left|\{\,(x,y) \in M^2 |\ x * y = y * x\}\right|}{|M|^2}. \tag{8}$$

Proposition 15. *If $\mathcal{A} = < A, \cdot, e, ^{-1} >$ and $\mathcal{B} = < B, \cdot, e, ^{-1} >$ are two finite polygroups, where $A = \{e, a_1, a_2, ..., a_{n-1}\}$ şi $B = \{e, b_1, b_2, ..., b_{m-1}\}$, with n, $m \in \mathbb{N}^*$, then the commutativity degree of polygroup $\mathcal{A}[\mathcal{B}]$, is*

$$d(\mathcal{A}[\mathcal{B}]) = \frac{n^2 d(\mathcal{A}) + m^2 d(\mathcal{B}) + 2\,(n-1)\,(m-1) - 1}{(n+m-1)^2}. \tag{9}$$

Proof. Let sets

$$\begin{aligned} A_1 &= \{\,(x,y) \in A^2 |\ x * y = y * x\}; \\ A_2 &= \{\,(x,y) \in B^2 |\ x * y = y * x\}; \\ A_3 &= \{\,(x,y) \in A \times B |\ x * y = y * x,\ \ x, y \neq e\}; \\ A_4 &= \{\,(x,y) \in B \times A |\ x * y = y * x, x, y \neq e\ \}. \end{aligned}$$

It is easy to observe that

$$\begin{aligned} A_1 \cap A_2 &= \{(e,e)\}, \\ A_i \cap A_j &= \emptyset,\ \forall\ (i,j) \neq (1,2)\,,\ i,j = \overline{1,4}. \end{aligned}$$

and

$$\{\,(x,y) \in M^2 |\ x * y = y * x\} = \bigcup_{i=1}^{4} A_i.$$

In conclusion,

$$d(\mathcal{A}[\mathcal{B}]) = \frac{n^2 d(\mathcal{A}) + m^2 d(\mathcal{B}) + 2\,(n-1)\,(m-1) - 1}{(n+m-1)^2}. \tag{10}$$

□

Example 16. *Let $\mathcal{P}_1 = < P_1, \cdot, e, ^{-1} >$ and $\mathcal{P}_2 = < P_2, \cdot, e, ^{-1} >$ be two polygroups, where $P_1 = \{e, a, b, c\}$ and $P_2 = \{e, a', b'\}$ thus :*

$$\mathcal{P}_1 : \begin{array}{ccccc} \cdot & e & a & b & c \\ e & e & a & b & c \\ a & a & a & P_1 & c \\ b & b & \{e,a,b\} & b & \{b,c\} \\ c & c & \{a,c\} & c & P_1 \end{array} ;\ \mathcal{P}_2 : \begin{array}{cccc} \cdot & e & a' & b' \\ e & e & a' & b' \\ a' & a' & \{e,b'\} & \{a',b'\} \\ b' & b' & \{a',b'\} & \{e,a'\} \end{array}.$$

The extension of polygroup $\mathcal{P}_1$ by polygroup $\mathcal{P}_2$, $\mathcal{P}_1[\mathcal{P}_2] = < M, *, e,^I >$ it is represents as follows

$$\begin{array}{ccccccc}
* & e & a & b & c & a' & b' \\
e & e & a & b & c & a' & b' \\
a & a & a & P_1 & c & a' & b' \\
b & b & \{e,a,b\} & b & \{b,c\} & a' & b' \\
c & c & \{a,c\} & c & P_1 & a' & b' \\
a' & a' & a' & a' & a' & \{e,b'\} \cup P_1 & \{a',b'\} \\
b' & b' & b' & b' & b' & \{a',b'\} & \{e,a'\} \cup P_1
\end{array}.$$

For $n = 4$, $m = 3$, $d(\mathcal{P}_1) = \frac{5}{8}$, $d(\mathcal{P}_2) = 1$ it is obatined:

$$d(\mathcal{P}_1[\mathcal{P}_2]) = \frac{4^2 \cdot \frac{5}{8} + 3^2 \cdot 1 + 2(4-1)(3-1) - 1}{(4+3-1)^2} = \frac{5}{6}.$$

We notice that $\frac{5}{6} > \frac{5}{8}$, so the result from group theory dosen't holds in polygroup theory.

Remark 17. *If $\mathcal{A}$ şi $\mathcal{B}$ are two commutative polygroups, then $d(\mathcal{A})$ =$d(\mathcal{B})$ =1 and*

$$d(\mathcal{A}[\mathcal{B}]) = \frac{n^2 + m^2 + 2\,(n-1)\,(m-1) - 1}{(n+m-1)^2} = 1.$$

So, $\mathcal{A}[\mathcal{B}]$ it is a commutative polygroup.

Remark 18. *The polygroup $P_G = G \cup \{a\}$, $a \notin G$, could be written as a extension of polygroup $\mathcal{A}$ =$<G, \cdot, e,^{-1}>$ by polygroup $\mathcal{B}$ =$<B, \cdot, e,^{-1}>$, where $B = \{e, a\}$, $a \notin G$ and " $\cdot$ " from $\mathcal{B}$ has the form:*

$\cdot$	e	a
e	e	a
a	a	$\{e,a\}$

. Applying the formula (9) *for $d(\mathcal{A})$ =$d(G)$, $m = 2$ and $d(\mathcal{B})$ =1, we obtain*

$$d(\mathcal{A}[\mathcal{B}]) = \frac{n^2 d(G) + 2^2 + 2\,(n-1) - 1}{(n+2-1)^2} = \frac{n^2 d(G) + 2n + 1}{(n+1)^2} = d(P_G).$$

Remark 19.

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2 d(\mathcal{A}) + m^2 d(\mathcal{B}) + 2\,(n-1)\,(m-1) - 1}{(n+m-1)^2} &= d(\mathcal{A}); \\
\lim_{m\to\infty} \frac{n^2 d(\mathcal{A}) + m^2 d(\mathcal{B}) + 2\,(n-1)\,(m-1) - 1}{(n+m-1)^2} &= d(\mathcal{B}).
\end{aligned}$$

We determine a boundaries for the extension $\mathcal{A}[\mathcal{B}]$, in the following.

Proposition 20. $min\{d(\mathcal{A}), d(\mathcal{B})\} \leq d(\mathcal{A}[\mathcal{B}]) \leq \frac{1+\max\{d(\mathcal{A}),d(\mathcal{B})\}}{2}$.

Proof. Let us suppose that $d(\mathcal{A}) \leq d(\mathcal{B})$. The other case is treated in a similar way.

$$d(\mathcal{A}[\mathcal{B}]) \geq \frac{n^2 d(\mathcal{A}) + m^2 d(\mathcal{A}) + 2\,(n-1)\,(m-1) - 1}{(n+m-1)^2}.$$

Equivalent with

$$(1 - d(\mathcal{A}))(2nm + 2n + 2m + 1) \geq 0.$$

Which is true, because $d(\mathcal{A}) \in (0, 1]$.

The next inequality becomes

$$\begin{aligned} d(\mathcal{A}[\mathcal{B}]) \quad &\leq \quad \frac{d(\mathcal{B})(n^2+m^2) + 2(n-1)(m-1) - 1}{(n+m-1)^2} \\ &= \quad \frac{\left(n^2+m^2\right)(d(\mathcal{B})-1)}{(n+m-1)^2} + 1. \end{aligned}$$

But,

$$\begin{aligned} \frac{\left(n^2+m^2\right)(d(\mathcal{B})-1)}{(n+m-1)^2} + 1 \quad &\leq \quad \frac{1+d(\mathcal{B})}{2} \Leftrightarrow \\ (d(\mathcal{B})-1)\left(\frac{n^2+m^2}{(n+m-1)^2} - \frac{1}{2}\right) \quad &\leq \quad 0, \end{aligned}$$

which is true. In conclusion,

$$min\{d(\mathcal{A}), d(\mathcal{B})\} \leq d(\mathcal{A}[\mathcal{B}]) \leq \frac{1+\max\{d(\mathcal{A}), d(\mathcal{B})\}}{2}.$$

□

4 On nilpotencity of $\mathcal{A}[\mathcal{B}]$

In this section, we propose to prove that if $\mathcal{A}$ and $\mathcal{B}$ are two nilpotent polygroups, then the extension of polygropus by polygroups, $\mathcal{A}[\mathcal{B}]$ is also a nilpotent polygroup. To prove this, we need some notions which appears in the book of B. Davvaz, [4].

Definition 21. *A polygroup* $< P, \cdot, e, ,^{-1} >$ *is said to be nilpotent, if* $l_n(P) \subseteq \omega_P$ *or equivalent* $l_n(P) \cdot \omega_P = \omega_P$*, for some integer* n*, where* $l_0(P) \cdot \omega_P = \omega_P$ *and*

$$l_{k+1}(P) = < \{h \in P |\ x \cdot y \cap h \cdot y \cdot x \neq \varnothing, \text{ such that } x \in l_k(P) \text{ and } y \in P\} > . \tag{11}$$

The smallest integer c such that $l_c(P) \cdot \omega_P = \omega_P$ *is called the nilpotencity class or for simplicity the class of* P.

Notice that $P = l_0(P) \supseteq l_1(P) \supseteq l_2(P) \supseteq$ ...that is $\{l_k(P)\}_{k \geq 0}$ is a decreasing sequence which we call it generalized descending central series.

Forwards, we find a connection between the heart of polygroups $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}[\mathcal{B}]$.

Proposition 22. *Let* $\mathcal{A}[\mathcal{B}] =< M, *, e,^I >$ *be the extension of polygroups* $\mathcal{A}$ *by* $\mathcal{B}$ *, where* $M = A \cup B$*,* $A \cap B = \{e\}$ *. The next relation, hold on*

$$\omega_{\mathcal{A}} \cup \omega_{\mathcal{B}} \subseteq \omega_{\mathcal{A}[\mathcal{B}]}. \tag{12}$$

Proof. Note that the neutral element e, is the same in all polygroups, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}[\mathcal{B}]$.

Let be $x \in \omega_{\mathcal{A}}$, follows that $x\beta^* e$, so there is $n \in \mathbb{N}$, $a_1, a_2, ..., a_n \in \mathcal{A}$ such that

$$\{x, e\} \subseteq \prod_{i=1}^{n} a_i. \tag{13}$$

Using the relation (7), it is observe that $x * y = x \cdot y$, for all x, $y \in \mathcal{A}$. So, the relation (12) could be written thus:

There is $n \in \mathbb{N}$, $a_1, a_2, ..., a_n \in \mathcal{A}[\mathcal{B}]$, such that $\{x, e\} \subseteq a_1 * a_2 * ... * a_n$ which implies $x \in \omega_{\mathcal{A}[\mathcal{B}]}$.

Now, if $x \in \omega_{\mathcal{B}}$, follows that $x\beta^* e$, so there is $m \in \mathbb{N}$, $b_1, b_2, ..., b_m \in \mathcal{B}$ such that

$$\{x, e\} \subseteq \prod_{i=1}^{n} b_i \subseteq b_1 * b_2 * ... * b_n, \tag{14}$$

if and only if $b_i \neq b_j^{-1}$, $\forall i$, $j \in \overline{1, n}$, so follows that $x \in \omega_{\mathcal{A}[\mathcal{B}]}$. If exists i, j such that $b_i = b_j^{-1}$, $\prod_{i=1}^{n} b_i \subseteq b_1 \cdot b_2 \cdot ...(b_i \cdot b_j \cup A) \cdot ... \cdot b_n$, so $x \in \omega_{\mathcal{A}[\mathcal{B}]}$. In conclusion $\omega_{\mathcal{A}} \cup \omega_{\mathcal{B}} \subseteq \omega_{\mathcal{A}[\mathcal{B}]}$. □

Proposition 23. *Let* $\mathcal{A} =< A, \cdot, e,^{-1} >$ *,* $\mathcal{B} =< B, \cdot, e,^{-1} >$ *be two polygroup. If* $\mathcal{A}[\mathcal{B}] =< M, *, e,^I >$ *is the extension of polygroups* $\mathcal{A}$ *by* $\mathcal{B}$ *, where* $M = A \cup B$*,* $A \cap B = \{e\}$ *, then*

$$l_k(\mathcal{A}[\mathcal{B}]) = l_k(\mathcal{A}) \cup l_k(\mathcal{B}). \tag{15}$$

Proof. We do the proof

$$l_k(\mathcal{A}) \cup l_k(\mathcal{B}) \subseteq l_k(\mathcal{A}[\mathcal{B}])$$

by induction on k. For $k = 0$, $A \cup B \subseteq A \cup B$, it is true. Now, suppose that $a \in l_{k+1}(\mathcal{A})$, so exists $x \in l_k(\mathcal{A})$, $y \in \mathcal{A}$, such that

$$x \cdot y \cap a \cdot y \cdot x \neq \emptyset.$$

Using the hypothesis induction, follows that $x \in l_k(\mathcal{A}[\mathcal{B}])$.

So, $a \in A \subset A \cup B$, $x \in l_k(\mathcal{A}[\mathcal{B}])$, $y \in A \cup B$ and

$$x * y \cap a * y * x \neq \emptyset.$$

In conclusion, $a \in l_{k+1}(\mathcal{A}[\mathcal{B}])$. If, $a \in l_{k+1}(\mathcal{B})$, exists $x \in l_k(\mathcal{B})$, $y \in \mathcal{B}$, such that $x \cdot y \cap a \cdot y \cdot x \neq \emptyset$. In a similar way, using the hypothesis induction, follows that $x \in l_k(\mathcal{A}[\mathcal{B}])$. So, we have two cases:

If, $y \neq x^{-1}$ and $y \neq a^{-1}$, the condition $x \cdot y \cap a \cdot y \cdot x \neq \emptyset$ becomes

$$x * y \cap a * y * x \neq \emptyset,$$

where $x \in l_k(\mathcal{A}[\mathcal{B}])$, $a \in A \cup B$, $y \in A \cup B$, follows that $a \in l_{k+1}(\mathcal{A}[\mathcal{B}])$.

If $y = x^{-1}$and $y \neq a^{-1}$, $x * y \cap a * y * x \neq \emptyset$ is equivalent with

$$(x \cdot y \cup A) \cap \left(\underset{c \in C}{\cup} a \cdot c \right) \neq \emptyset,$$

where $C = y \cdot x \cup A$, because $x \cdot y \cap a \cdot y \cdot x \neq \emptyset$. So, $a \in l_{k+1}(\mathcal{A}[\mathcal{B}])$. The other cases are treated in a similar way.

Now, like above, using the induction method, we do the proof

$$l_k(\mathcal{A}[\mathcal{B}]) \subseteq l_k(\mathcal{A}) \cup l_k(\mathcal{B}).$$

For $k = 0$, $A \cup B \subseteq A \cup B$. If $a \in l_{k+1}(\mathcal{A}[\mathcal{B}])$, then $a \in A \cup B$ and exists $x \in l_k(\mathcal{A}[\mathcal{B}])$, $y \in A \cup B$ such that

$$x * y \cap a * y * x \neq \emptyset. \tag{16}$$

Using the hypothesis induction, follows that $x \in l_k(\mathcal{A})$ or $x \in l_k(\mathcal{B})$.

If $a \in A$, we choose $x \in l_k(\mathcal{A})$ and $y \in A$ such that the condition (15) becomes $x \cdot y \cap a \cdot y \cdot x \neq \emptyset$, results $a \in l_{k+1}(\mathcal{A})$.

If $a \in B$ we choose $x \in l_k(\mathcal{B})$ and $y \in B$, such that the condition (15) becomes

$$\begin{cases} x \cdot y \cap a \cdot y \cdot x \neq \emptyset, & y \neq a^{-1} \neq x^{-1} \\ (x \cdot y \cup A) \cap \left(\underset{c \in C}{\cup} a \cdot c \right) \neq \emptyset, & y = x^{-1} \\ x \cdot y \cap \left(\underset{d \in D}{\cup} d \cdot x \right) \neq \emptyset, & y = a^{-1} \end{cases}, \tag{17}$$

where $D = a \cdot y \cup A$. From the relations given by (17), it is obtained that $x \cdot y \cap a \cdot y \cdot x \neq \emptyset$. $\square$

Proposition 24. *If $\mathcal{A}$ and $\mathcal{B}$ are nilpotent polygroups, then the extension of polygroups, $\mathcal{A}[\mathcal{B}]$ is also a nilpotent polygroup.*

Proof. $\mathcal{A}$ is a nilpotent polygroups, so there exists $k_1 \in \mathbb{N}^*$ such that $l_{k_1}(\mathcal{A}) \subseteq \omega_{\mathcal{A}}$. $\mathcal{B}$ is a nilpotent polygroups, so there exists $k_2 \in \mathbb{N}^*$ such that $l_{k_2}(\mathcal{B}) \subseteq \omega_{\mathcal{B}}$.

$$l_{k_1}(\mathcal{A}) \cup l_{k_2}(\mathcal{B}) \subseteq \omega_{\mathcal{A}} \cup \omega_{\mathcal{B}} \subseteq \omega_{\mathcal{A}[\mathcal{B}]} \tag{18}$$

Let $k = max(k_1, k_2)$ and $\{l_k(P)\}_{k\geq 0}$ is a decreasing sequence. We have $l_k(\mathcal{A}) \subseteq l_{k_1}(\mathcal{A})$ and $l_k(\mathcal{B}) \subseteq l_{k_2}(\mathcal{B})$. Using the Proposition 23, follows that

$$l_k(\mathcal{A}[\mathcal{B}]) \subseteq \omega_{\mathcal{A}[\mathcal{B}]}.$$

So, $\mathcal{A}[\mathcal{B}]$ is a nilpotent polygroup. □

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